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Some Type of Improper Fractional Integral

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we evaluate some type of improper fractional integral. A new multiplication of fractional analytic functions plays an important role in this paper. The main methods we used are integration by parts for fractional calculus and fractional L'Hospital's rule. At the same time, some examples are given to illustrate our result. In fact, our result is a generalization of the classical calculus result.**

Keywords: **Jumarie type of R-L fractional calculus, improper fractional integral, new multiplication, fractional analytic functions, integration by parts, fractional L'Hospital's rule.**

I. INTRODUCTION

During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, mechanics, biology, control theory, electrical engineering, viscoelasticity, and so on [1-8].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [9-14]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional calculus, we solve the following improper fractional integral:

$$\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes[E_{\alpha}(tx^{\alpha})-1]^{\otimes-1}\right]. \quad (1)$$

Where $0<\alpha\leq 1$ and $t>0$. Integration by parts for fractional calculus, fractional L'Hospital's rule, and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our result is a generalization of the result of ordinary calculus.

II. DEFINITIONS AND PROPERTIES

Firstly, the fractional calculus used in this paper is introduced below.

Definition 2.1 ([15])**:** Suppose that $0<\alpha\leq 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \quad (2)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \quad (3)$$

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where $\Gamma(\ )$ is the gamma function.

In the following, we introduce some properties of Jumarie type of fractional derivative.

Proposition 2.2 ([16])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)[(x-x_0)^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{5}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([17])**:** Assume that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([18])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{7}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{8}$$

In other words,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{9}$$

Definition 2.5 ([18])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{13}$$

Definition 2.6 ([19])**:** Let $0 < \alpha \le 1$, and x be a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

Definition 2.7: Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

Theorem 2.8 (integration by parts for fractional calculus) ([20])**:** *Assume that* $0 < \alpha \le 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, *then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = \left[f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{15}$$

Theorem 2.9 (fractional L'Hospital's rule) ([21])**:** *Assume that* $0 < \alpha \le 1$, c *is a real number, and* $f_\alpha(x^\alpha), g_\alpha(x^\alpha)$, $[g_\alpha(x^\alpha)]^{\otimes -1}$ *are* α*-fractional analytic functions at* $x = c$. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, *and* $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. *Suppose that* $\lim_{x\to c} f_\alpha(x^\alpha)\otimes[g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)]\otimes\left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$ *exist,* $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \neq 0$. *Then*

$$\lim_{x\to c} f_\alpha(x^\alpha)\otimes[g_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)]\otimes\left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}. \tag{16}$$

III. MAIN RESULT AND EXAMPLES

In this section, we obtain the main result in this paper and we provide some examples to illustrate our result.

Theorem 3.1: *Let* $0 < \alpha \le 1$, *and* $t > 0$. *Then the improper* α*-fractional integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes[E_\alpha(tx^\alpha)-1]^{\otimes -1}\right] = \frac{\pi^2}{6t^2}. \tag{17}$$

Proof

$$\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes[E_\alpha(tx^\alpha)-1]^{\otimes -1}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes E_\alpha(-tx^\alpha)\otimes[1-E_\alpha(-tx^\alpha)]^{\otimes -1}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes E_\alpha(-tx^\alpha)\otimes\sum_{n=1}^{\infty}[E_\alpha(-tx^\alpha)]^{\otimes(n-1)}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\sum_{n=1}^{\infty}[E_\alpha(-tx^\alpha)]^{\otimes n}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\sum_{n=1}^{\infty}E_\alpha(-tnx^\alpha)\right]$$

$$= \sum_{n=1}^{\infty}\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes E_\alpha(-tnx^\alpha)\right]$$

$$= \sum_{n=1}^{\infty}\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\left({}_0D_x^\alpha\right)\left[\frac{1}{-tn}E_\alpha(-tnx^\alpha)\right]\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\frac{1}{-tn}E_\alpha(-tnx^\alpha)\Big|_{x=0}^{x=+\infty} - \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{-tn}E_\alpha(-tnx^\alpha)\right]\right]$$

(by integration by parts for fractional calculus)

$$= \sum_{n=1}^{\infty}\left[-\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{-tn}E_\alpha(-tnx^\alpha)\right]\right]$$

(by fractional L'Hospital's rule)

$$= \sum_{n=1}^{\infty} \left[\frac{1}{tn} \left({}_0I_{+\infty}^{\alpha} \right) [E_\alpha(-tnx^\alpha)] \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{tn} \left[\frac{1}{-tn} E_\alpha(-tnx^\alpha) \right] \Big|_{x=0}^{x=+\infty} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{tn} \cdot \frac{1}{tn} \right]$$

$$= \frac{1}{t^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{t^2} \cdot \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{6t^2}.$$ Q.e.d.

Example 3.2: Let $0 < \alpha \leq 1$. Then by Theorem 3.1, the improper α-fractional integrals

$$\left({}_0I_{+\infty}^{\alpha} \right) \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes [E_\alpha(x^\alpha) - 1]^{\otimes -1} \right] = \frac{\pi^2}{6}, \tag{18}$$

$$\left({}_0I_{+\infty}^{\alpha} \right) \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes [E_\alpha(3x^\alpha) - 1]^{\otimes -1} \right] = \frac{\pi^2}{54}, \tag{19}$$

$$\left({}_0I_{+\infty}^{\alpha} \right) \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left[E_\alpha\left(\sqrt{2} x^\alpha\right) - 1 \right]^{\otimes -1} \right] = \frac{\pi^2}{12}, \tag{20}$$

$$\left({}_0I_{+\infty}^{\alpha} \right) \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left[E_\alpha\left(\frac{1}{2} x^\alpha\right) - 1 \right]^{\otimes -1} \right] = \frac{2\pi^2}{3}. \tag{21}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we solve some type of improper fractional integral mainly using integration by parts for fractional calculus and fractional L'Hospital's rule. A new multiplication of fractional analytic functions plays an important role in this article. On the other hand, some examples are provided to illustrate our result. In fact, our result is a generalization of the result of traditional calculus. In the future, we will continue to use these methods to study the problems in engineering mathematics and fractional differential equations.

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